LAPLACE TRANSFORM

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INTRODUCTION TO LAPLACE TRANSFORM

- The basic result we got from previous chapter is that the response of an LTI system is given by convolution of the input and the impulse response of the system.
- Now we present an alternative representation for signals and LTI systems.
- The Laplace transform is introduced to represent continuous-time signals in the s-domain (s is a complex variable), and the concept of the system function for a continuous-time LTI system is described.

INTRODUCTION TO LAPLACE TRANSFORM

- Fourier transform enable us to understand the bahavior of a system in the frequency domain by allowing a signal $x(t)$ to be represented as a continuous sum of a complex exponentials.
- Fourier Transform is restricted to only thosse functions for which the Fourier transform exists

LAPLACE TRANSFORM

- That for a continuous-time LTI system with impulse response h(t), the output y(t) of the system to the complex exponential input of the form e^{st} is

$$y(t) = T\{e^{st}\} = H(s)e^{st}$$

Where

$$H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st} dt$$

LAPLACE TRANSFORM

- The function H(s) is reffered to as the Laplace transform of h(t).
- For general continuous-time signal x(t), the Laplace transform X(s) is defined as:

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st} dt$$

- The variable s is generally complex valued and is expressed as:

$$s = \sigma + j\omega$$

- The Lapalace transform is often called the bilateral (or two-sided)
- While Laplace transform in to the unilateral (or one-sided) Laplace transform is defined as

$$X_1(s) = \int_{0^-}^{\infty} x(t)\, e^{-st} dt$$

LAPLACE TRANSFORM

- Equation

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st} dt$$

- Is sometimes considered an operator that transforms a signal x(t) into a function X(s) symbolically represented by

$$X(s) = \mathcal{L}\{x(t)\}$$

- and the signal x(t) and its Laplace transform X(s) are said to form a Laplace transform pair denoted as

$$x(t) \leftrightarrow X(s)$$

REGION OF CONVERGENCE

- The range of values of the complex variables s for which the Laplace transform converges is called the region of convergence (ROC).
- To illustrate the Laplace transform and the associated ROC let us consider some example:

- Consider the signal

$$x(t) = e^{-at}u(t) \;\; ; a \; real$$

- Find the laplace transform of $x(t)$

REGION OF CONVERGENCE

$x(t) = e^{-at}u(t)$

Using equation

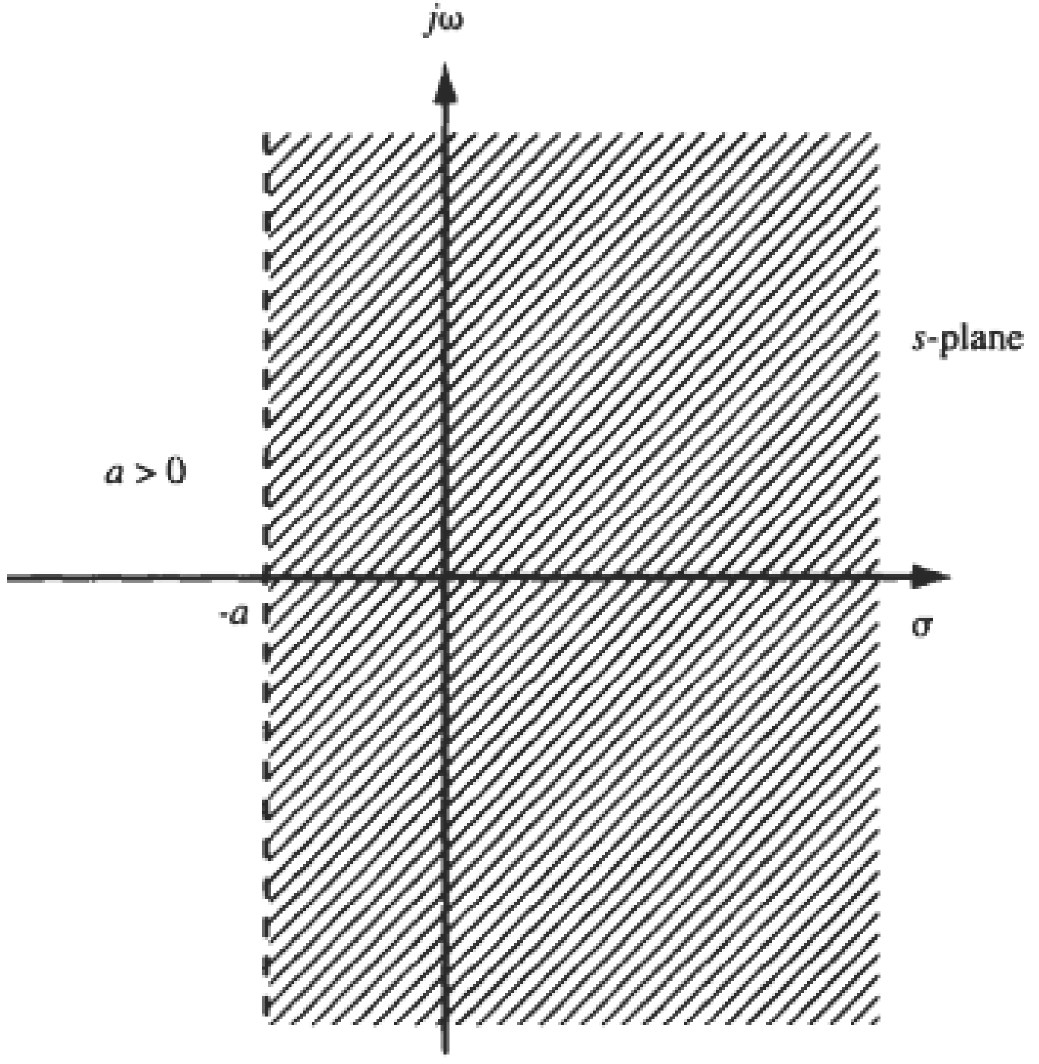
$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st} dt$$

$$X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)\, e^{-st} dt = \int_{0^+}^{\infty} e^{-(s+a)t} dt$$

$$= -\frac{1}{s+a} e^{-(s+a)t}\Big|_{0^+}^{\infty} = \frac{1}{s+a} \qquad Re(s) > -a$$

Because $\lim_{t\to\infty} e^{-(s+a)t} = 0$ **only if** $Re(s+a) > 0\ or\ Re(s) > -a$

REGION OF CONVERGENCE



REGION OF CONVERGENCE

- Consider the signal

$$x(t) = -e^{-at}u(-t) \;\; ; a \, real$$

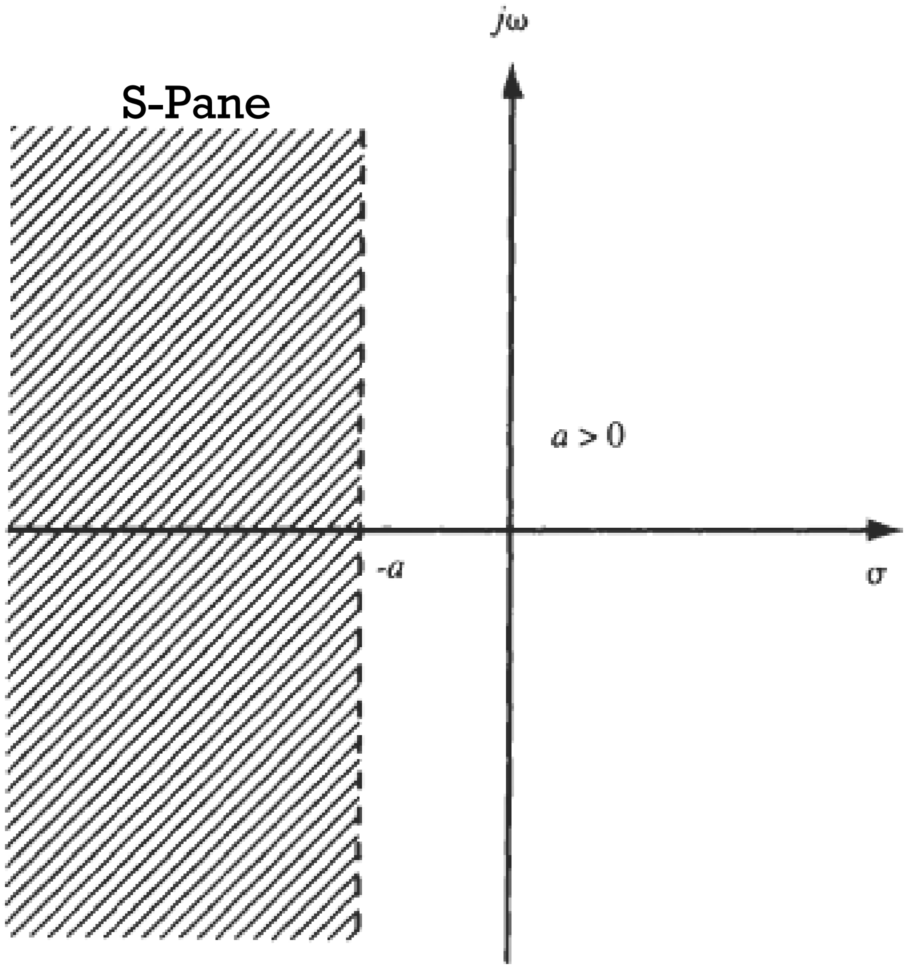
- The laplace transform X(s) is:

$$X(s) = \int_{-\infty}^{\infty} -e^{-at}u(-t)\, e^{-st} dt = -\int_{-\infty}^{0} e^{-(s+a)t} dt$$

$$= \frac{1}{s+a} \qquad Re(s) < -a$$

REGION OF CONVERGENCE

S-Pane



POLES AND ZEROS

- Usually, X(s) will be a rational function in s, that is,

$$X(s) = \frac{a_0 s^m + a_1 s^{m-1} + \cdots + a_m}{b_0 s^n + b_1 s^{n-1} + \cdots + b_n} = \frac{a_0}{b_0} \frac{(s - z_1) \cdots (s - z_m)}{(s - p_1) \cdots (s - p_n)}$$

zeros

poles

- The roots of the numerator polynomial, z_k are called the zeros of X(s) because X(s)= 0 for those values of s.
- Similarly, the roots of the denominator polynomial, p_k are called the poles of X(s) because X(s) is infinite for those values of s
- Traditionally, an " x " is used to indicate each pole location and an " *0 " is used to* indicate each zero.

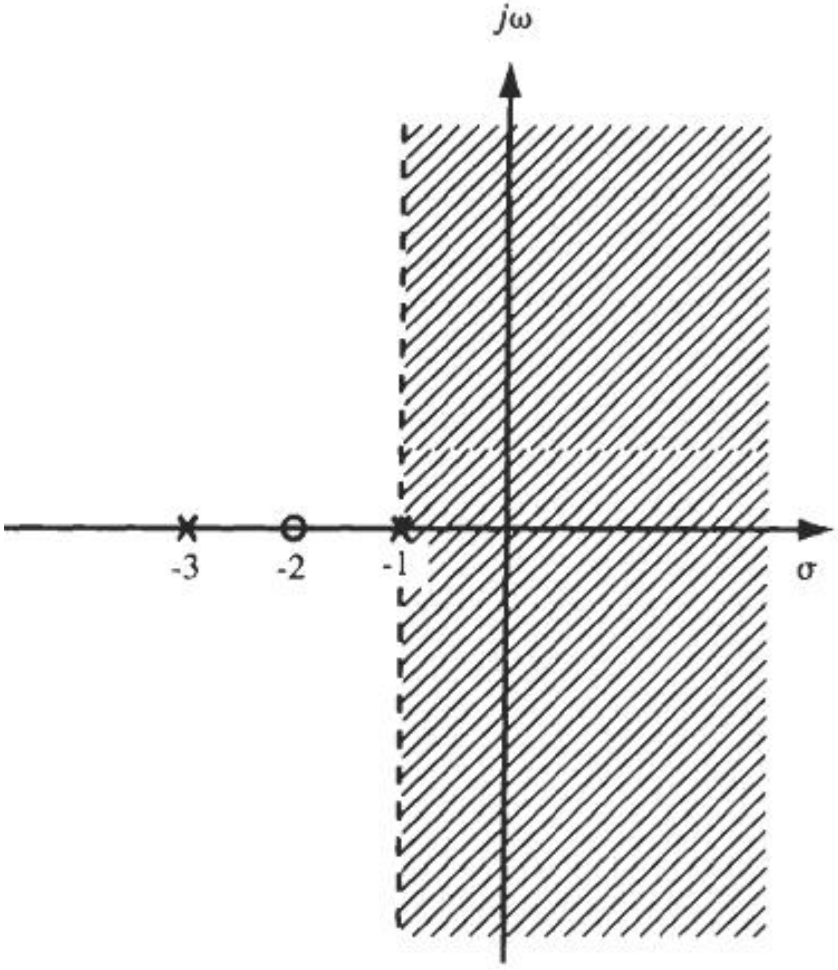
POLES AND ZEROS

$$X(s) = \frac{2s + 4}{s^2 + 4s + 3}$$

$$= 2\frac{s + 2}{(s + 1)(s + 3)} \qquad \mathrm{Re}(s) > -1$$

X(s) has one zero at s = - 2 and two poles at s = - 1 and s = - 3 with scale factor 2.

POLES AND ZEROS



LAPLACE TRANSFORM FOR COMMON SIGNALS

$$\mathscr{L}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-st}\, dt = 1 \qquad \text{all } s$$

$$\mathscr{L}[u(t)] = \int_{-\infty}^{\infty} u(t) e^{-st}\, dt = \int_{0^+}^{\infty} e^{-st}\, dt$$

$$= -\frac{1}{s} e^{-st} \Big|_{0^+}^{\infty} = \frac{1}{s} \qquad \mathrm{Re}(s) > 0$$

EXAMPLE

- Find the Laplace transform of the following signals
- $x(t) = e^{-t}u(t) + e^{-2t}u(t)$
- $x(t) = \delta(t) - \frac{3}{4}e^{-t}u(t) + \frac{1}{2}e^{-t}u(t)$

Find the Laplace transform $X(s)$ and sketch the pole-zero plot with the ROC for the following signals $x(t)$:

(a) $x(t) = e^{-2t}u(t) + e^{-3t}u(t)$

(b) $x(t) = e^{-3t}u(t) + e^{2t}u(-t)$

(c) $x(t) = e^{2t}u(t) + e^{-3t}u(-t)$

PROPERTIES OF ROC

Property 1: The ROC does not contain any poles.

Property 2: If $x(t)$ is a *finite-duration* signal, that is, $x(t) = 0$ except in a finite interval $t_1 \le t \le t_2$ $(-\infty < t_1$ and $t_2 < \infty)$, then the ROC is the entire s-plane except possibly $s = 0$ or $s = \infty$.

Property 3: If $x(t)$ is a *right-sided* signal, that is, $x(t) = 0$ for $t < t_1 < \infty$, then the ROC is of the form

$$\mathrm{Re}(s) > \sigma_{max}$$

where σ_{max} equals the maximum real part of any of the poles of $X(s)$. Thus, the ROC is a half-plane to the right of the vertical line $\mathrm{Re}(s) = \sigma_{max}$ in the s-plane and thus to the right of all of the poles of $X(s)$.

PROPERTIES OF ROC

Property 4: If $x(t)$ is a *left-sided* signal, that is, $x(t) = 0$ for $t > t_2 > -\infty$, then the ROC is of the form

$$\text{Re}(s) < \sigma_{\min}$$

where $\sigma_{\min}$ equals the minimum real part of any of the poles of $X(s)$. Thus, the ROC is a half-plane to the left of the vertical line $\text{Re}(s) = \sigma_{\min}$ in the s-plane and thus to the left of all of the poles of $X(s)$.

Property 5: If $x(t)$ is a *two-sided* signal, that is, $x(t)$ is an infinite-duration signal that is neither right-sided nor left-sided, then the ROC is of the form

$$\sigma_1 < \text{Re}(s) < \sigma_2$$

where σ_1 and σ_2 are the real parts of the two poles of $X(s)$. Thus, the ROC is a vertical strip in the s-plane between the vertical lines $\text{Re}(s) = \sigma_1$ and $\text{Re}(s) = \sigma_2$.

LAPLACE TRANSOFM PAIRS

Table 3-1 Some Laplace Transforms Pairs

$x(t)$	$X(s)$	ROC
$\delta(t)$	1	All s
$u(t)$	$\frac{1}{s}$	$\text{Re}(s) > 0$
$-u(-t)$	$\frac{1}{s}$	$\text{Re}(s) < 0$
$tu(t)$	$\frac{1}{s^2}$	$\text{Re}(s) > 0$
$t^k u(t)$	$\frac{k!}{s^{k+1}}$	$\text{Re}(s) > 0$

LAPLACE TRANSOFM PAIRS

$e^{-at}u(t)$	$\dfrac{1}{s+a}$	$\mathrm{Re}(s) > -\mathrm{Re}(a)$
$-e^{-at}u(-t)$	$\dfrac{1}{s+a}$	$\mathrm{Re}(s) < -\mathrm{Re}(a)$
$te^{-at}u(t)$	$\dfrac{1}{(s+a)^2}$	$\mathrm{Re}(s) > -\mathrm{Re}(a)$
$-te^{-at}u(-t)$	$\dfrac{1}{(s+a)^2}$	$\mathrm{Re}(s) < -\mathrm{Re}(a)$

LAPLACE TRANSOFM PAIRS

$\cos \omega_0 t u(t)$	$\dfrac{s}{s^2+\omega_0^2}$	$\mathrm{Re}(s) > 0$
$\sin \omega_0 t u(t)$	$\dfrac{\omega_0}{s^2+\omega_0^2}$	$\mathrm{Re}(s) > 0$
$e^{-at} \cos \omega_0 t u(t)$	$\dfrac{s+a}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}(s) > -\mathrm{Re}(a)$
$e^{-at} \sin \omega_0 t u(t)$	$\dfrac{\omega_0}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}(s) > -\mathrm{Re}(a)$

LAPLACE TRANSFORM PROPERTIES

LINIARITY

If

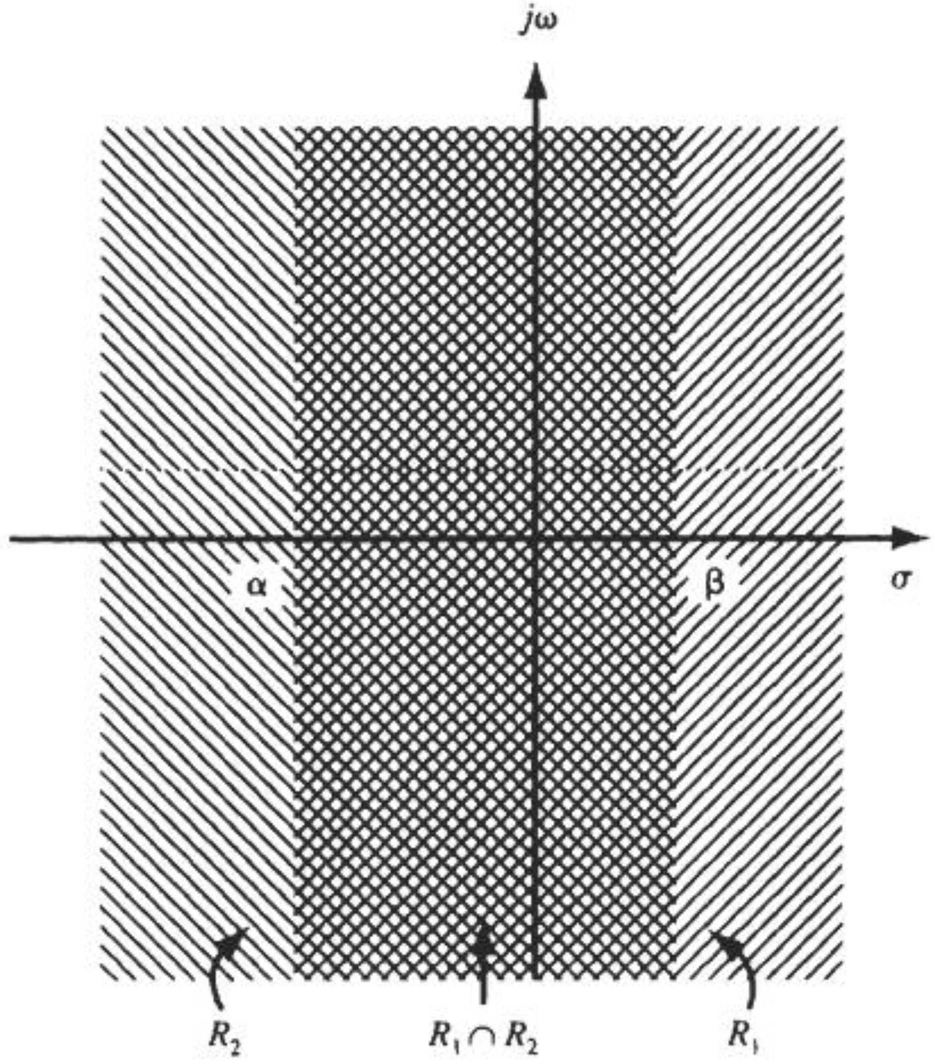
$$x_1(t) \longleftrightarrow X_1(s) \qquad \text{ROC} = R_1$$

$$x_2(t) \longleftrightarrow X_2(s) \qquad \text{ROC} = R_2$$

Then $$a_1 x_1(t) + a_2 x_2(t) \longleftrightarrow a_1 X_1(s) + a_2 X_2(s) \qquad R' \supset R_1 \cap R_2$$

- The set notation $A \supset B$ means that set A contains set B
- while $A \cap B$ denotes the intersection of sets A and B,
- that is, the set containing all elements in both A and B.
- This indicates that the ROC of the resultant Laplace transform is at least as large as the region in common between R1 and R2.
- Usually we have simply $R' = R1 \cap R2$

LINIARITY



LINIARITY

Linearity

If $F_1(s)$ and $F_2(s)$ are, respectively, the Laplace transforms of $f_1(t)$ and $f_2(t)$, then

$$\mathcal{L}[a_1 f_1(t) + a_2 f_2(t)] = a_1 F_1(s) + a_2 F_2(s) \tag{15.7}$$

where a_1 and a_2 are constants. Equation 15.7 expresses the linearity property of the Laplace transform. The proof of Eq. (15.7) follows readily from the definition of the Laplace transform in Eq. (15.1).

For example, by the linearity property in Eq. (15.7), we may write

$$\mathcal{L}[\cos wt] = \mathcal{L}\left[\frac{1}{2}(e^{j\omega t} + e^{-j\omega t})\right] = \frac{1}{2}\mathcal{L}[e^{j\omega t}] + \frac{1}{2}\mathcal{L}[e^{-j\omega t}] \tag{15.8}$$

But from Example 15.1(b), $\mathcal{L}[e^{-at}] = 1/(s+a)$. Hence,

$$\mathcal{L}[\cos wt] = \frac{1}{2}\left(\frac{1}{s - j\omega} + \frac{1}{s + j\omega}\right) = \frac{s}{s^2 + \omega^2} \tag{15.9}$$

TIME SHIFTING

If

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

then

$$x(t - t_0) \leftrightarrow e^{-st_0} X(s) \qquad R' = R$$

This indicates that the ROCs before and after the time-shift operation are the same.

TIME SHIFTING

Time Shift

$$\mathcal{L}[f(t-a)u(t-a)] = e^{-as}F(s)$$

As an example, we know from Eq. (15.9) that

$$\mathcal{L}[\cos \omega t] = \frac{s}{s^2 + \omega^2}$$

Using the time-shift property in Eq. (15.17),

$$\mathcal{L}[\cos \omega(t-a)u(t-a)] = e^{-as}\frac{s}{s^2 + \omega^2}$$

SHIFTING IN THE S-DOMAIN

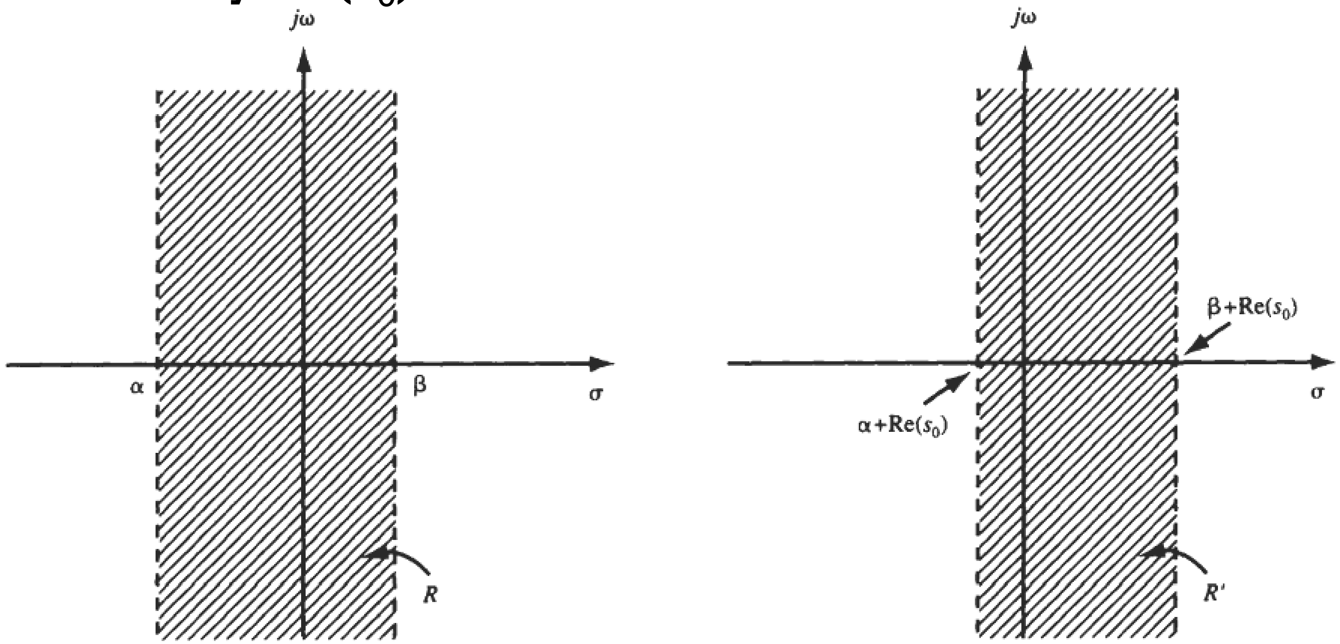
If

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

then

$$e^{s_0 t} x(t) \leftrightarrow X(s - s_0) \qquad R' = R + \text{Re}(s_0)$$

- This indicates that the ROC associated with $X(s-s_0)$ is that of $X(s)$ shifted by $\text{Re}(s_0)$.



SHIFTING IN S-DOMAIN

$$\mathcal{L}[e^{-at} f(t)] = F(s + a)$$

As an example, we know that

$$\cos \omega t \iff \frac{s}{s^2 + \omega^2}$$

and (15.20)

$$\sin \omega t \iff \frac{\omega}{s^2 + \omega^2}$$

Using the shift property in Eq. (15.19), we obtain the Laplace transform of the damped sine and damped cosine functions as

$$\mathcal{L}[e^{-at} \cos \omega t] = \frac{s + a}{(s + a)^2 + \omega^2} \tag{15.21a}$$

$$\mathcal{L}[e^{-at} \sin \omega t] = \frac{\omega}{(s + a)^2 + \omega^2} \tag{15.21b}$$

TIME SCALING

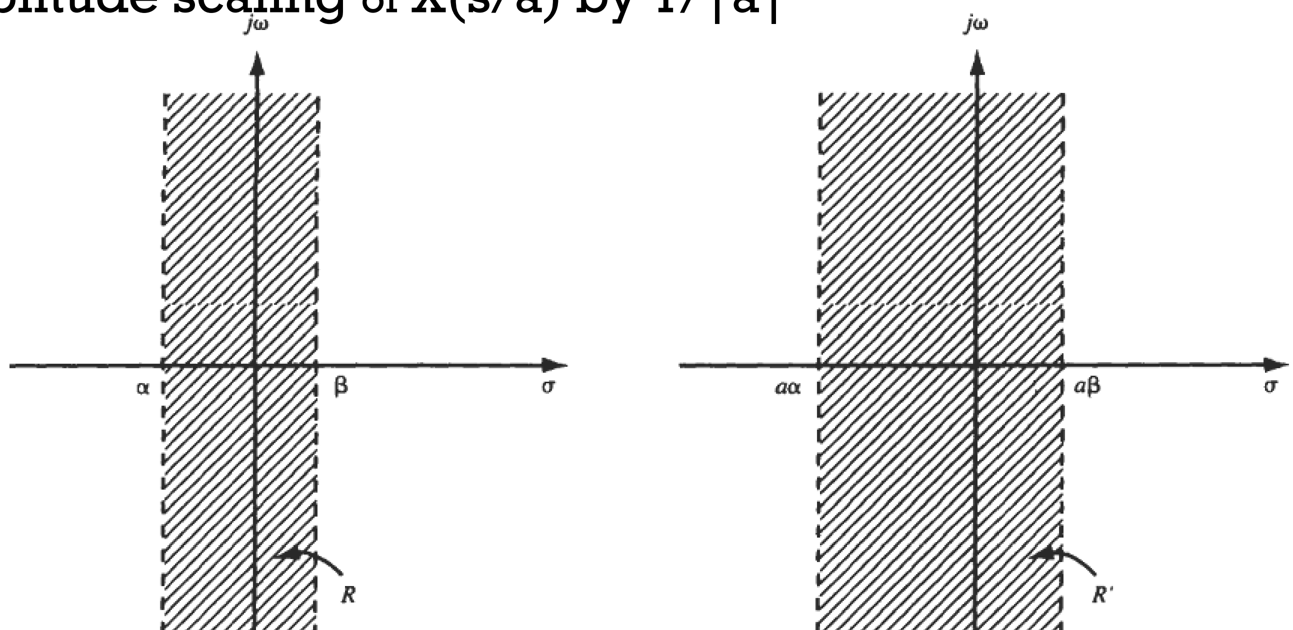
If

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

then

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{s}{a}\right) \qquad R' = aR$$

- This indicates that scaling the time variable t by the factor a causes an inverse scaling of the variable s by 1/a as well as amplitude scaling of X(s/a) by 1/|a|



TIME SCALING

Scaling

If $F(s)$ is the Laplace transform of $f(t)$,

$$\mathcal{L}[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$$

For example, we know from Example 15.2 that

$$\mathcal{L}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2} \tag{15.13}$$

Using the scaling property in Eq. (15.12),

$$\mathcal{L}[\sin 2\omega t] = \frac{1}{2} \frac{\omega}{(s/2)^2 + \omega^2} = \frac{2\omega}{s^2 + 4\omega^2} \tag{15.14}$$

which may also be obtained from Eq. (15.13) by replacing ω with 2ω.

TIME REVERSAL

If

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

then

$$x(-t) \leftrightarrow X(-s) \qquad R' = -R$$

- Time reversal of x(t) produces a reversal of both the σ and the $j\omega$ axes in the s-plane.

SUMMARY OF LAPLACE TRANSFORM PROPERTY

Property	Signal	Transform	ROC
	$x(t)$	$X(s)$	R
	$x_1(t)$	$X_1(s)$	R_1
	$x_2(t)$	$X_2(s)$	R_2
Linearity	$a_1x_1(t) + a_2x_2(t)$	$a_1X_1(s) + a_2X_2(s)$	$R' \supset R_1 \cap R_2$
Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	$R' = R$
Shifting in s	$e^{s_0t}x(t)$	$X(s - s_0)$	$R' = R + \mathrm{Re}(s_0)$
Time scaling	$x(at)$	$\frac{1}{\|a\|}X(s)$	$R' = aR$
Time reversal	$x(-t)$	$X(-s)$	$R' = -R$
Differentiation in t	$\frac{dx(t)}{dt}$	$sX(s)$	$R' \supset R$
Differentiation in s	$-tx(t)$	$\frac{dX(s)}{ds}$	$R' = R$
Integration	$\int_{-\infty}^{t} x(\tau)\,d\tau$	$\frac{1}{s}X(s)$	$R' \supset R \cap \{\mathrm{Re}(s) > 0\}$
Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	$R' \supset R_1 \cap R_2$

INVERSE LAPLACE TRANSFORM

INVERSION FORMULA

- Inversion of the Laplace transform to find the signal x(t) from its Laplace transform X(s) is called the inverse Laplace transform, symbolically denoted as

$$x(t) = \mathcal{L}^{-1}\{X(s)\}$$

- Generally the formula is

$$x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}ds$$

In this integral, the real c is to be selected such that if the ROC of $X(s)$ is $\sigma_1 < \mathrm{Re}(s) < \sigma_2$, then $\sigma_1 < c < \sigma_2$. The evaluation of this inverse Laplace transform integral requires an understanding of complex variable theory.

TABLES OF LAPLACE TRANSFORM PAIRS

- In the second method for the inversion of X(s), we attempt to express X(s) as a sum

$$X(s) = X_1(s) + \cdots + X_n(s)$$

- where $X_1(s), \ldots, X_n(s)$ are functions with known inverse transforms $x_1(t), \ldots, x_n(t)$. From the linearity property it follows that

$$x(t) = x_1(t) + \cdots + x_n(t)$$

PARTIAL FRACTION EXPANSION

- If X(s) is a rational function, that is, of the form

$$X(s) = \frac{N(s)}{D(s)} = k\frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)}$$

- a simple technique based on partial-fraction expansion can be used for the inversion of X(s)

PARTIAL FRACTION EXPANSION

a) When X(s) is a proper rational function, that is, when m < n:

1. **Simple Pole Case:**

- If all poles of X(s), that is, all zeros of D(s), are simple (or distinct), then X(s) can be written as

$$X(s) = \frac{c_1}{s - p_1} + \cdots + \frac{c_n}{s - p_n}$$

- where coefficients c_k are given by

$$c_k = (s - p_k)X(s)\big|_{s = p_k}$$

PARTIAL FRACTION EXPANSION

2. Multiple Pole Case

- If D(s) has multiple roots, that is, if it contains factors of the form $(s - p_i)^r$,
- we say that p_i is the multiple pole of X(s) with multiplicity r.
- Then the expansion of X(s) will consist of terms of the form

$$\frac{\lambda_1}{s - p_i} + \frac{\lambda_2}{(s - p_i)^2} + \cdots + \frac{\lambda_r}{(s - p_i)^r}$$

- Where

$$\lambda_{r-k} = \frac{1}{k!} \frac{d^k}{ds^k} \left[(s - p_i)^r X(s) \right] \Big|_{s = p_i}$$

PARTIAL FRACTION EXPANSION

b) When X(s) is an improper rational function, when $\mathrm{m} \geq n$:

c) If $\mathrm{m} \geq n$, by long division we can write X(s) in the form

$$X(s) = \frac{N(s)}{D(s)} = Q(s) + \frac{R(s)}{D(s)}$$

EXAMPLES

- Find the inverse Laplace Transform of the following X(s)

(a) $X(s) = \dfrac{2s+4}{s^2+4s+3}, \ \mathrm{Re}(s) > -1$

(b) $X(s) = \dfrac{2s+4}{s^2+4s+3}, \ \mathrm{Re}(s) < -3$

(c) $X(s) = \dfrac{2s+4}{s^2+4s+3}, \ -3 < \mathrm{Re}(s) < -1$

(d) $X(s) = \dfrac{s^2+2s+5}{(s+3)(s+5)^2} \qquad \mathrm{Re}(s) > -3$

$X(s) = \dfrac{2s+1}{s+2}, \ \mathrm{Re}(s) > -2$

$X(s) = \dfrac{s^2+6s+7}{s^2+3s+2}, \ \mathrm{Re}(s) > -1$

$X(s) = \dfrac{s^3+2s^2+6}{s^2+3s}, \ \mathrm{Re}(s) > 0$

A, B, C

$$X(s) = \frac{2s+4}{s^2+4s+3} = 2\frac{s+2}{(s+1)(s+3)}$$

$$= \frac{c_1}{s+1} + \frac{c_2}{s+3}$$

$$c_1 = (s+1)X(s)\big|_{s=-1} \qquad c_2 = (s+3)X(s)\big|_{s=-3}$$

$$= 2\frac{s+2}{s+3}\bigg|_{s=-1} = 1 \qquad = 2\frac{s+2}{s+1}\bigg|_{s=-3} = 1$$

$$X(s) = \frac{1}{s+1} + \frac{1}{s+3} \qquad x(t) = e^{-t}u(t) + e^{-3t}u(t) = (e^{-t} + e^{-3t})u(t)$$

D

$$X(s) = \frac{c_1}{s+3} + \frac{\lambda_1}{s+5} + \frac{\lambda_2}{(s+5)^2}$$

$$c_1 = (s+3)X(s)\big|_{s=-3}$$

$$= \frac{s^2+2s+5}{(s+5)^2}\bigg|_{s=-3} = 2$$

$$\lambda_2 = (s+5)^2 X(s)\big|_{s=-5}$$

$$\lambda_1 = \frac{d}{ds}\left[(s+5)^2 X(s)\right]\bigg|_{s=-5}$$

$$= \frac{d}{ds}\left[\frac{s^2+2s+5}{s+3}\right]\bigg|_{s=-5} = \frac{s^2+6s+1}{(s+3)^2}\bigg|_{s=-5} = -1$$

$$X(s) = \frac{2}{s+3} - \frac{1}{s+5} - \frac{10}{(s+5)^2}$$

$$x(t) = 2e^{-3t}u(t) - e^{-5t}u(t) - 10te^{-5t}u(t)$$

$$= [2e^{-3t} - e^{-5t} - 10te^{-5t}]u(t)$$

D

Note that there is a simpler way of finding λ_1 without resorting to differentiation. This is shown as follows: First find c_1 and λ_2 according to the regular procedure. Then substituting the values of c_1 and λ_2 into Eq. (*3.84*), we obtain

$$\frac{s^2+2s+5}{(s+3)(s+5)^2} = \frac{2}{s+3} + \frac{\lambda_1}{s+5} - \frac{10}{(s+5)^2}$$

Setting $s = 0$ on both sides of the above expression, we have

$$\frac{5}{75} = \frac{2}{3} + \frac{\lambda_1}{5} - \frac{10}{25}$$

from which we obtain $\lambda_1 = -1$.

E

$$X(s) = \frac{2s+1}{s+2} = \frac{2(s+2)-3}{s+2} = 2 - \frac{3}{s+2}$$

$\mathrm{Re}(s) > -2$, $x(t)$ is a right-sided signal

$$x(t) = 2\delta(t) - 3e^{-2t}u(t)$$

F

$$X(s) = \frac{s^2 + 6s + 7}{s^2 + 3s + 2}$$

$$= 1 + \frac{3s + 5}{s^2 + 3s + 2}$$

$$= 1 + \frac{3s + 5}{(s + 1)(s + 2)}$$

$$X_1(s) = \frac{3s + 5}{(s + 1)(s + 2)} = \frac{c_1}{s + 1} + \frac{c_2}{s + 2}$$

$$c_1 = (s + 1)X_1(s)\big|_{s=-1} = \frac{3s + 5}{s + 2}\bigg|_{s=-1} = 2$$

$$c_2 = (s + 2)X_1(s)\big|_{s=-2} = \frac{3s + 5}{s + 1}\bigg|_{s=-2} = 1$$

$$X(s) = 1 + \frac{2}{s + 1} + \frac{1}{s + 2}$$

$\mathrm{Re}(s) > -1$. Thus, $x(t)$ is a right-sided signal

$$x(t) = \delta(t) + (2e^{-t} + e^{-2t})u(t)$$

G

$$X(s) = \frac{s^3 + 2s^2 + 6}{s^2 + 3s} = s - 1 + \frac{3s + 6}{s(s + 3)}$$

$$X_1(s) = \frac{3s + 6}{s(s + 3)} = \frac{c_1}{s} + \frac{c_2}{s + 3}$$

$$c_1 = sX_1(s)\big|_{s=0} = \left.\frac{3s + 6}{s + 3}\right|_{s=0} = 2$$

$$c_2 = (s + 3)X_1(s)\big|_{s=-3} = \left.\frac{3s + 6}{s}\right|_{s=-3} = 1$$

$$X(s) = s - 1 + \frac{2}{s} + \frac{1}{s + 3} \qquad x(t) = \delta'(t) - \delta(t) + (2 + e^{-3t})u(t)$$

THE SYSTEM FUNCTION

THE SYSTEM FUNCTION

$$y(t) = x(t) * h(t)$$

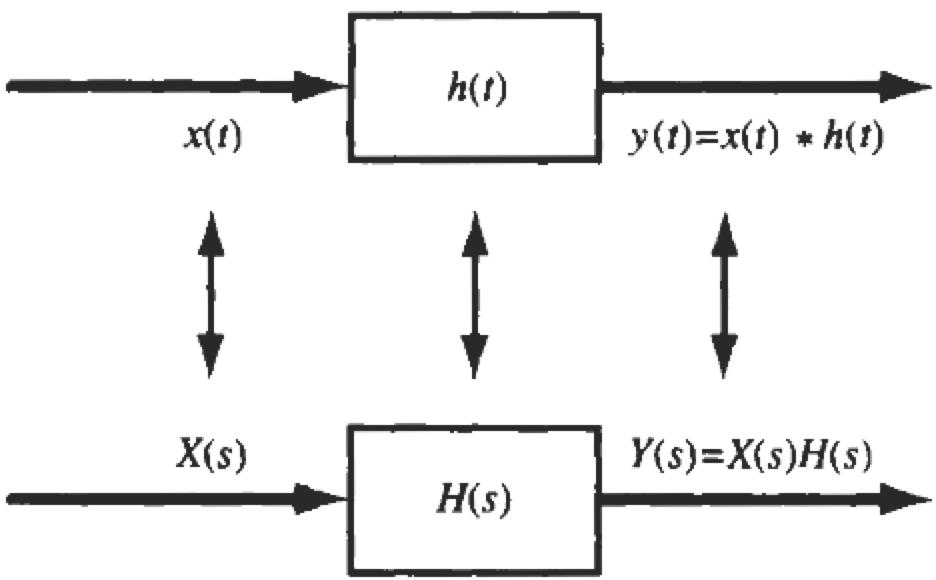
Applying the convolution property

$$Y(s) = X(s)H(s)$$

$$H(s) = \frac{Y(s)}{X(s)}$$

CHARACTERIZATION OF LTI SYSTEM

- Many properties of continuous-time LTI systems can be closely associated with the characteristics of H(s) in the s-plane and in particular with the pole locations and the ROC.



CAUSALITY

- For a causal continuous-time LTI system, we have

$$h(t) = 0 \qquad t < 0$$

- Since h(t) is a right-sided signal, the corresponding requirement on H(s) is that the ROC of H(s) must be of the form

$$\mathrm{Re}(s) > \sigma_{\max}$$

- That is, the ROC is the region in the s-plane to the right of all of the system poles.
- Similarly, if the system is anticausal, then

$$h(t) = 0 \qquad t > 0$$

$$\mathrm{Re}(s) < \sigma_{\min}$$

STABILITY

- A continuous-time LTI system is BIB0 stable if and only if

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

$$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st}\, dt \qquad \text{Let } s = j\omega. \text{ Then}$$

$$|H(j\omega)| = \left| \int_{-\infty}^{\infty} h(t) e^{-j\omega t}\, dt \right| \le \int_{-\infty}^{\infty} \left| h(t) e^{-j\omega t} \right| dt$$

$$= \int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

- The corresponding requirement on H(s) is that the ROC of H(s) contains the jω-axis (that is, s = jω)

CAUSAL AND STABLE SYSTEMS

- If the system is both causal and stable, then
- all the poles of H(s) must lie in the left half of the s-plane; that is, they all have negative real parts because the ROC is of the form

$$\mathrm{Re}(s) > \sigma_{\max}$$

- and since the jω axis is included in the ROC, we must have a

$$\sigma_{\max} < 0.$$

EXAMPLES

The output y(t) of a continuous time LTI system is found to be $2e^{-3t}u(t)$ when the input $x(t)$ is $u(t)$

- Find the impulse response h(t) of the system
- Find the output y(t) when the input x(t) is $e^{-t}u(t)$

Answer $y(t) = (-e^{-t} + 3e^{-3t})u(t)$

- x(t)=u(t) → $X(s)=\frac{1}{s}$ Re(s)>0
- $y(t)=2e^{-3t}u(t)$ → $Y(s)=\frac{2}{s+3}$ Re(s)>-3
- $H(s)=\frac{Y(s)}{X(s)} = \frac{2s}{s+3}$ → $H(s)=\frac{2(s+3)-6}{s+3} = 2 - \frac{6}{s+3}$
- $h(t) = 2\delta(t) - 6e^{-3t}u(t)$

EXAMPLES

- Consider a continuous time LTI system for which the input x(t) and output y(t) are related by:

$$y''(t) + y'(t) - 2y(t) = x(t)$$

- Find the system function H(s)
- Determine the impulse response (h(t)) for each of the following cases:
  - The system is causal
  - The system is stable
  - The system is neither causal nor stable

A

$$y''(t) + y'(t) - 2y(t) = x(t)$$

$$s^2 Y(s) + sY(s) - 2Y(s) = X(s)$$

$$(s^2 + s - 2)Y(s) = X(s)$$

$$H(s) = \frac{Y(s)}{X(s)}$$

$$= \frac{1}{s^2 + s - 2} = \frac{1}{(s+2)(s-1)}$$

B

$$H(s) = \frac{1}{(s+2)(s-1)} = -\frac{1}{3}\frac{1}{s+2} + \frac{1}{3}\frac{1}{s-1}$$

Causal

ROC of $H(s)$ is $\mathrm{Re}(s) > 1$.

$$h(t) = -\tfrac{1}{3}(e^{-2t} - e^{t})u(t)$$

Stable

ROC of $H(s)$ is $-2 < \mathrm{Re}(s) < 1$.

$$h(t) = -\tfrac{1}{3}e^{-2t}u(t) - \tfrac{1}{3}e^{t}u(-t)$$

Not Causal Not Stable

ROC of $H(s)$ is $\mathrm{Re}(s) < -2$.

$$h(t) = \tfrac{1}{3}e^{-2t}u(-t) - \tfrac{1}{3}e^{t}u(-t)$$