FOURIER TRANSFORM

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2016

INTRODUCTION

- convert time-domain signals into frequency-domain (or *spectral) representations.*
- In 1807, Jean Baptiste Joseph Fourier submitted a paper using trigonometric series to represent "any" periodic signals
- But Lagrange rejected it!!
- In 1822, Fourier published a book "The Analytical Theory of Heat"
- He also claimed that "any" periodic signal could be represented by Fourier series.
- He however obtained a representation for aperiodic signals i.e., Fourier integral or transform
- Fourier did not actually contribute to the mathematical theory of Fourier series

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FOURIER SERIES REPRESENTATION OF PERIODIC SIGNALS

PERIODIC SIGNALS

- a continuous-time signal x(t) to be periodic if there is a positive nonzero value of T for which:

$$x(t+T) = x(t) \quad for\ \ all\ t$$

- The fundamental period T_0 of $x(t)$ is the smallest positive value of T
- $f_0 = \frac{1}{T_0}$ is referred to as the fundamental frequency
- Two basic examples of periodic signals are the real sinusoidal signal

$$x(t) = \cos(\omega_0 t + \phi)$$

- And the complex exponential signal

$$x(t) = e^{j\omega t}$$

where $\omega_0 = 2\pi/T_0 = 2\pi f_0$ is called the *fundamental angular frequency*.

COMPLEX EXPONENTIAL FOURIER SERIES REPRESENTATION

- The complex exponential Fourier series representation of a periodic signal x(t) with fundamental period T_0 is given by

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \quad ; \quad \omega_0 = \frac{2\pi}{T_0}$$

- *where c, are known as the complex Fourier coefficients and are given by*

$$c_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt$$

- Where $\int_{T_0}$ denotes the integral over any one period and 0 to *To or -To/2 to T0/2 is* commonly used for the integration.

COMPLEX EXPONENTIAL FOURIER SERIES REPRESENTATION

- Set k = 0, then:

$$c_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt$$

- which indicates that c_0 equals the average value of x(t) over a period.

- When x(t) is real, then it follows that:

$$c_{-k} = c_k^*$$

- where the asterisk indicates the complex conjugate.

TRIGONOMETRIC FOURIER SERIES

- The trigonometric Fourier series representation of a periodic signal x(t) with fundamental period T_0 is given by

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\omega_0 t + b_k \sin k\omega_0 t) \qquad \omega_0 = \frac{2\pi}{T_0}$$

- where a_k and b_k are the Fourier coefficients given by

$$a_k = \frac{2}{T_0} \int_{T_0} x(t) \cos k\omega_0 t \, dt$$

$$b_k = \frac{2}{T_0} \int_{T_0} x(t) \sin k\omega_0 t \, dt$$

EVEN AND ODD SIGNALS

- If a periodic signal x(t) is even, then $b_k = 0$ and its Fourier series contains only cosine terms:

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\omega_0 t \qquad \omega_0 = \frac{2\pi}{T_0}$$

- If x(t) is odd, then $a_k = 0$ and its Fourier series contains only sine terms:

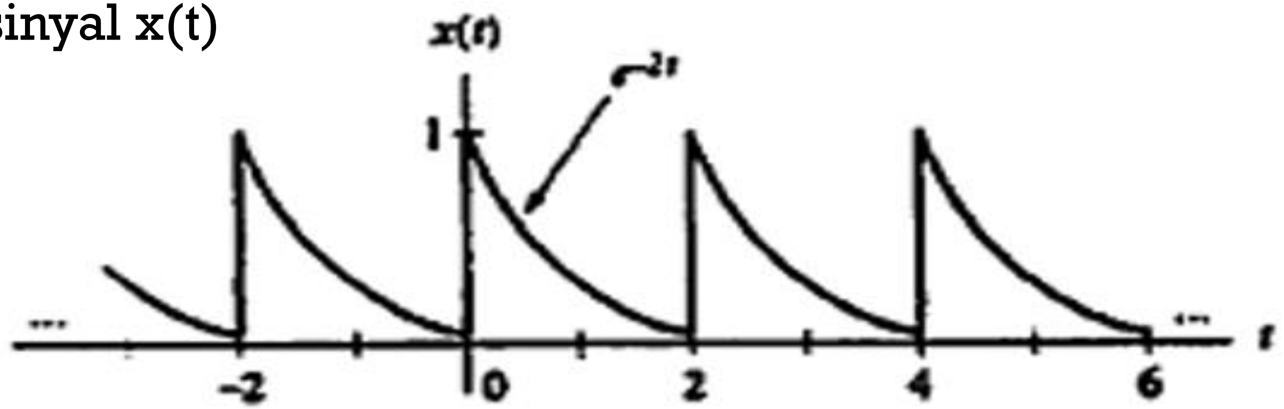
$$x(t) = \sum_{k=1}^{\infty} b_k \sin k\omega_0 t \qquad \omega_0 = \frac{2\pi}{T_0}$$

PERSAMAAN PENTING

- **Sinyal Sinusoid Kompleks**
- **x(t) =** $e^{jk\omega_o t} = \; cosk\omega_o t$+ j sin$k\omega_o t$
- **Konjugate**
- x*(t) = $e^{-jk\omega_o t}$= $cosk\omega_o t$- j sin$k\omega_o t$
- **Rumus Euler**

$$\cos k\omega_o t = \frac{e^{jk\omega_o t} + e^{-jk\omega_o t}}{2} \qquad \sin k\omega_o t = \frac{e^{jk\omega_o t} - e^{-jk\omega_o t}}{2j}$$

Tentukan koefisien DF untuk sinyal x(t)



Perioda x(t) adalah T=2, $\omega_o = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$

$x(t) = e^{-2t}, 0 \le t \le 2$; $x(t) = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_o t} dt$

$x(t) = \frac{1}{2}\int_0^2 e^{-2t}\, e^{-jk\omega_o t} dt$

$$X[k] = \frac{1}{2}\int_0^2 e^{-2t}\, e^{-jk\pi t} dt$$

$$X[k] = \frac{1}{2}\int_0^2 e^{-(2+jk\pi)t}\, dt$$

$$X[k] = \frac{-1}{2(2+jk\pi)} e^{-(2+jk\pi)t} \Big|_0^2$$

$$X[k] = \frac{-1}{4+jk2\pi}\left(e^{-(2+jk\pi)2} - e^{-(2+jk\pi)0}\right)$$

$$X[k] = \frac{-1}{4+jk2\pi}\left(e^{-(2+jk\pi)2} - e^{0}\right)$$

$$X[k] = \frac{-1}{4+jk2\pi}\left(e^{-4-2jk\pi} - 1\right)$$

$$X[k] = \frac{-1}{4+jk2\pi}\left(e^{-4}e^{-2jk\pi} - 1\right)$$

$$X[k] = \frac{-1}{4+jk2\pi}\left(e^{-4}(1) - 1\right)$$

$$e^{-2jk\pi} = \cos 2k\pi - j\sin k\pi$$

$$e^{-2jk\pi} = 1 - 0$$

$$e^{-2jk\pi} = 1\,, untuk\; 0 \le k \le \sim$$

Tentukan representasi Deret Fourier Sinyal :

$x(t) = 3\cos\left(\frac{\pi}{2}t + \frac{\pi}{4}\right)$ memakai metoda inspeksi.

$\omega_o = \frac{\pi}{2}$, $T = \frac{2\pi}{\omega_o} = \frac{2\pi}{\frac{\pi}{2}} = 4$ **(Perioda fundamental/Perioda Dasar)**

$$x(t) = 3\cos\left(\frac{\pi}{2}t + \frac{\pi}{4}\right) = 3\,\frac{e^{j\left(\frac{\pi}{2}t+\frac{\pi}{4}\right)} + e^{-j\left(\frac{\pi}{2}t+\frac{\pi}{4}\right)}}{2}$$

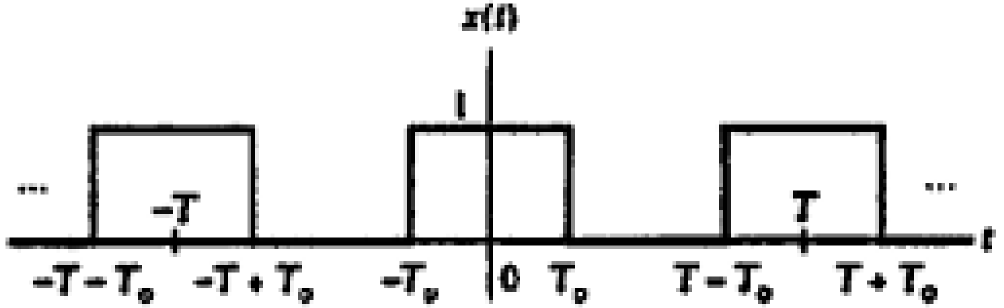
$$x(t) = \frac{3}{2}e^{j\frac{\pi}{4}}e^{j\frac{\pi}{2}t} + \frac{3}{2}e^{-j\frac{\pi}{4}}e^{-j\frac{\pi}{2}t}$$

Maka $X[k] = \begin{cases} \frac{3}{2}e^{-j\frac{\pi}{4}}, \boldsymbol{untuk\ k = -1} \\ \frac{3}{2}e^{j\frac{\pi}{4}}, \boldsymbol{untuk\ k = 1} \\ \boldsymbol{0\ , nilai\ lain} \end{cases}$

$$x(t) = \sum_{k=-\sim}^{\sim} X[k]\, e^{jk\omega_o t}$$

DF Gelombang Segi-Empat (1)

- Tentukan representasi DF gelombang segi-empat:



- Solusi: Perioda adalah T, dimana $\omega_0 = \frac{2\pi}{T}$.
- Karena sinyal $x(t)$ simetri genap, maka perhitungan:

$$X[k] = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt$$

- Integrasi dilakukan pada selang: $-\frac{T}{2} \leq t \leq \frac{T}{2}$.

$$X[k] = \frac{1}{T}\int_{-T_0}^{T_0} e^{-jk\omega_0 t}dt = \frac{-1}{Tjk\omega_0}e^{-jk\omega_0 t}\Big|_{-T_0}^{T_0}, \qquad k \neq 0$$

DF Gelombang Segi-Empat (2)

- $X[k] = \frac{2}{Tk\omega_0}\left(\frac{e^{jk\omega_0 T_0} - e^{-jk\omega_0 T_0}}{2j}\right), \qquad k \neq 0.$

  $X[k] = \frac{2sin(k\omega_0 T_0)}{Tk\omega_0}, \quad k \neq 0.$

  Untuk $k = 0$:

$$X[0] = \frac{1}{T}\int_{-T_0}^{T_0} dt = \frac{2T_0}{T}$$

  Memakai aturan L'Hopital's: $\lim_{k \to 0} \frac{2sin(k\omega_0 T_0)}{Tk\omega_0} = \frac{2T_0}{T}.$

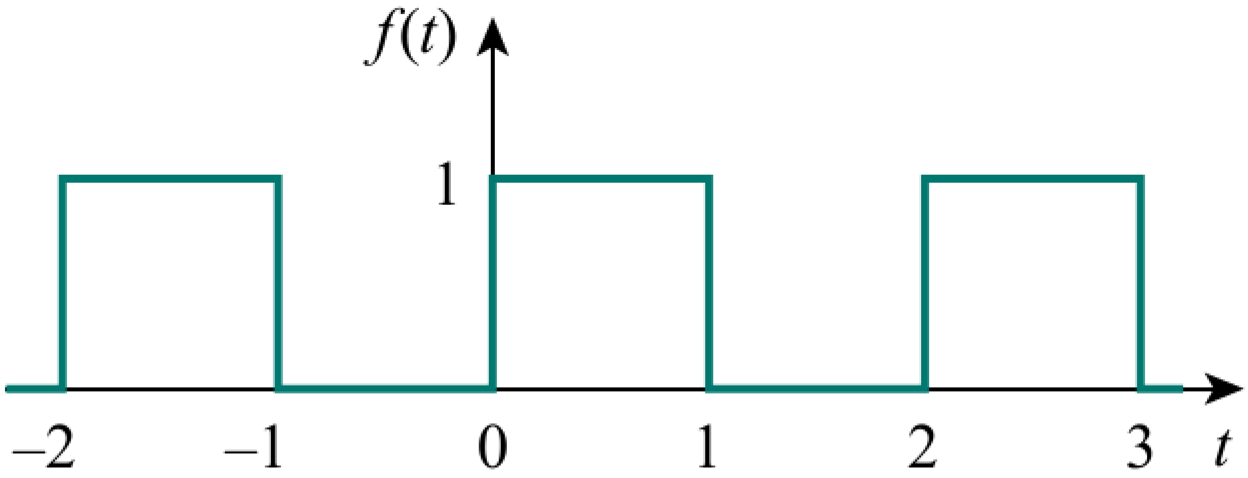
- $X[k]$ nilainya riil, dengan memakai $\omega_0 = \frac{2\pi}{T}$

  Diperoleh:

$$X[k] = \frac{2sin\left(k\frac{2\pi}{T}T_0\right)}{k2\pi}$$

CONTOH

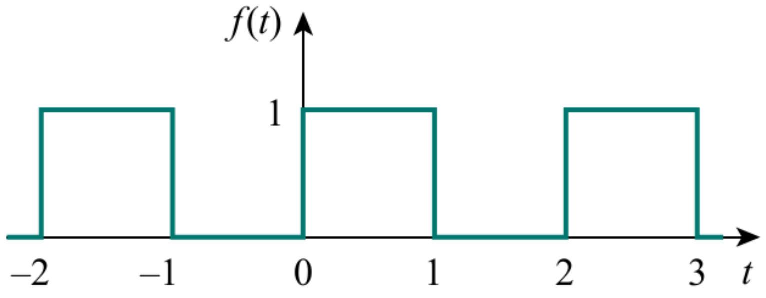
- Tentukan deret Fourier dari bentuk gelombang periodik berikut ini



- Ekspresi deret Fourier diberikan berikut ini

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

- Tujuan kita adalah mendapatkan koefisien deret a_0, a_n, dan b_n
- Dari gambar diperoleh
  - T=2
  - $\omega_0 = 2\pi/T = \pi$



$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{2}\left[\int_0^1 1\,dt + \int_1^2 0\,dt\right] = \frac{1}{2}t\Big|_0^1 = \frac{1}{2}$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\,dt$$

$$= \frac{2}{2}\left[\int_0^1 1\cos n\pi t\,dt + \int_1^2 0\cos n\pi t\,dt\right]$$

$$= \frac{1}{n\pi}\sin n\pi t\,\Big|_0^1 = \frac{1}{n\pi}\sin n\pi = 0$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t\,dt$$

$$= \frac{2}{2}\left[\int_0^1 1\sin n\pi t\,dt + \int_1^2 0\sin n\pi t\,dt\right]$$

$$= -\frac{1}{n\pi}\cos n\pi t\,\Big|_0^1$$

$$= -\frac{1}{n\pi}(\cos n\pi - 1), \qquad \cos n\pi = (-1)^n$$

$$= \frac{1}{n\pi}[1-(-1)^n] = \begin{cases} \dfrac{2}{n\pi}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

- Dengan mensubtitusikan koefisien diperoleh:

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sin \pi t + \frac{2}{3\pi}\sin 3\pi t + \frac{2}{5\pi}\sin 5\pi t + \cdots$$

- Dapat diringkas sebagai:

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{n}\sin n\pi t, \qquad n = 2k - 1$$

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

PERTIMBANGAN SIMETRI (1)

- Jika sinyal periodik tersebut diketahui simetrik genap

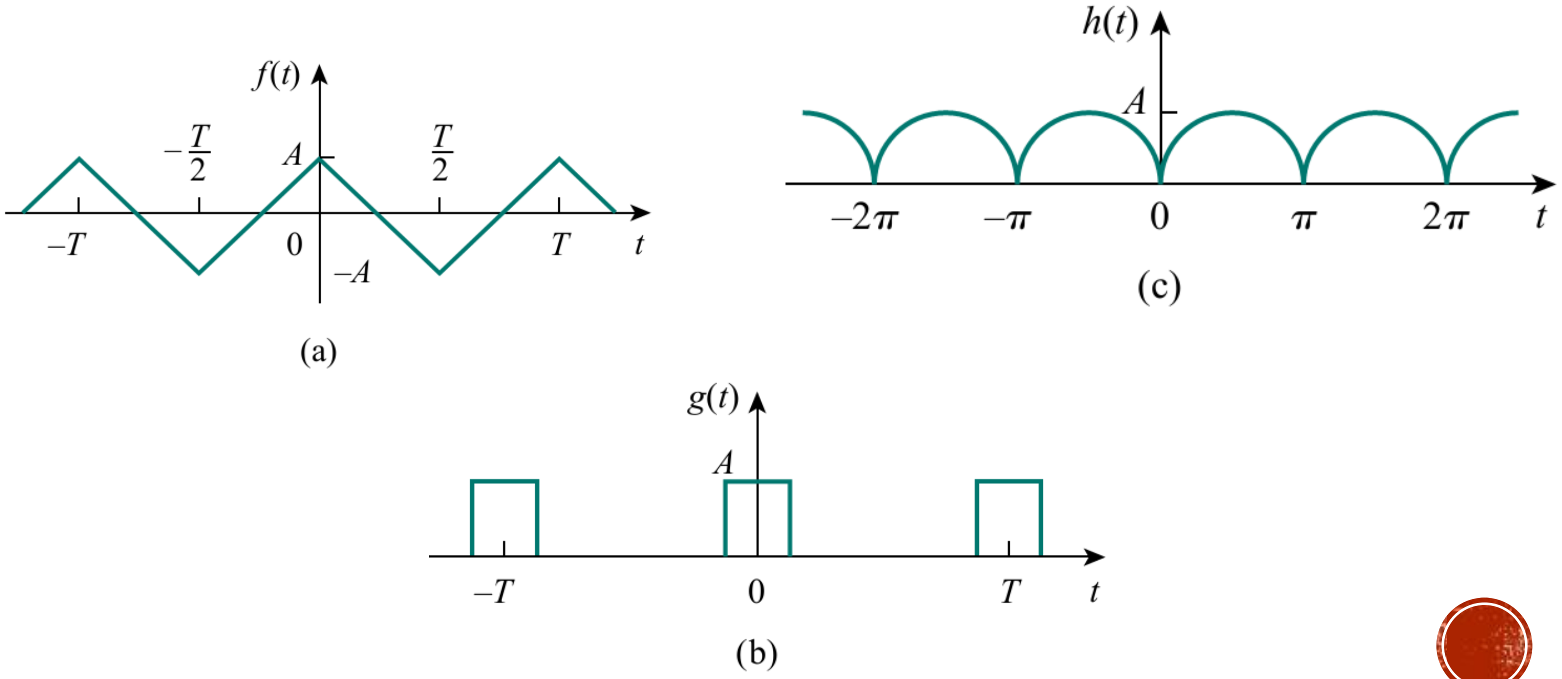
$$f(t) = f(-t)$$

- maka diperoleh

$$a_0 = \frac{2}{T}\int_0^{T/2} f(t)\,dt$$

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega_0 t\,dt$$

$$b_n = 0$$



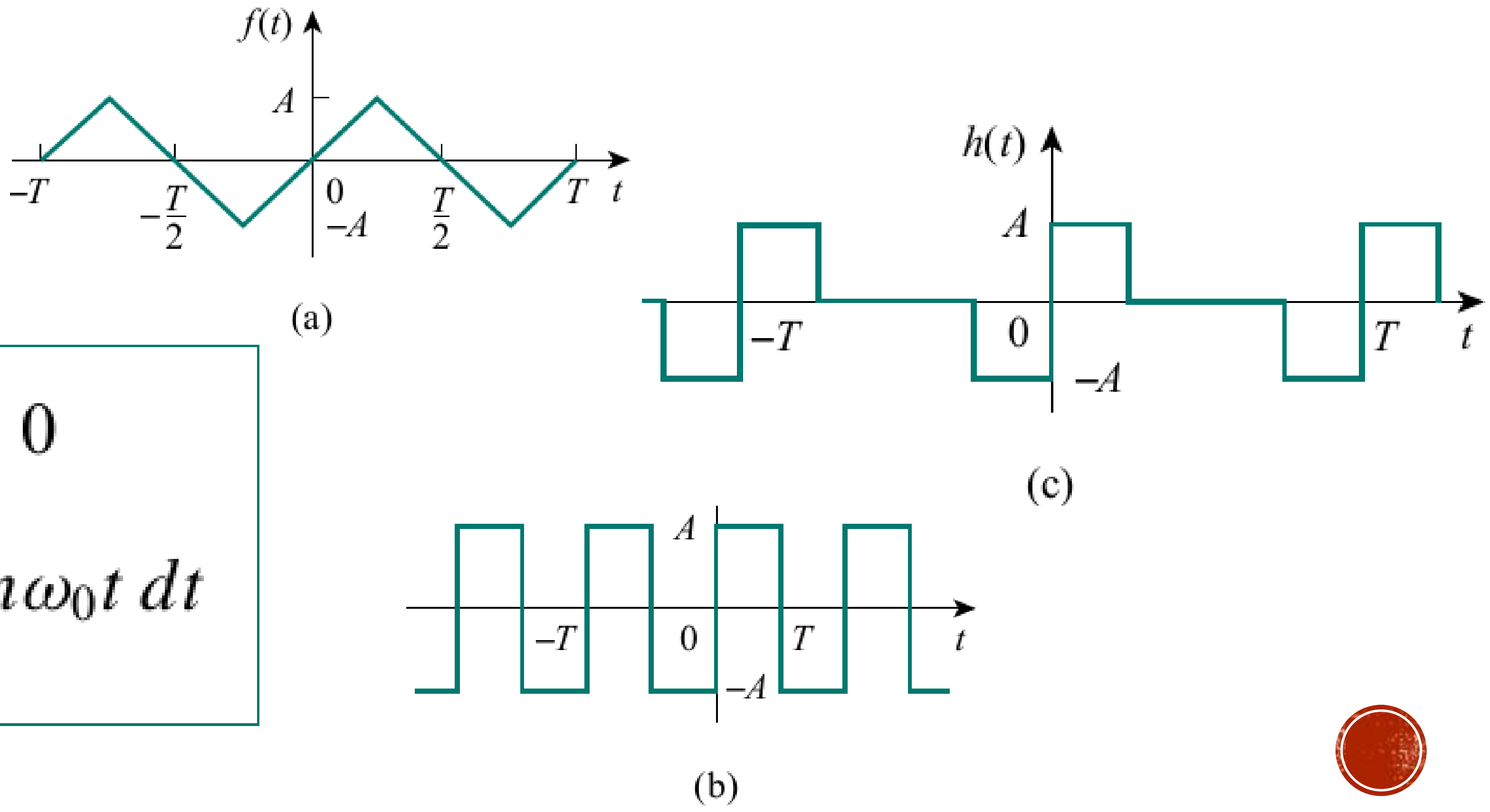
$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

PERTIMBANGAN SIMETRI (2)

- Jika sinyal periodik tersebut diketahui simetrik ganjil

$$f(-t) = -f(t)$$

- maka diperoleh



$$a_0 = 0, \qquad a_n = 0$$

$$b_n = \frac{4}{T}\int_0^{T/2} f(t) \sin n\omega_0 t \, dt$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

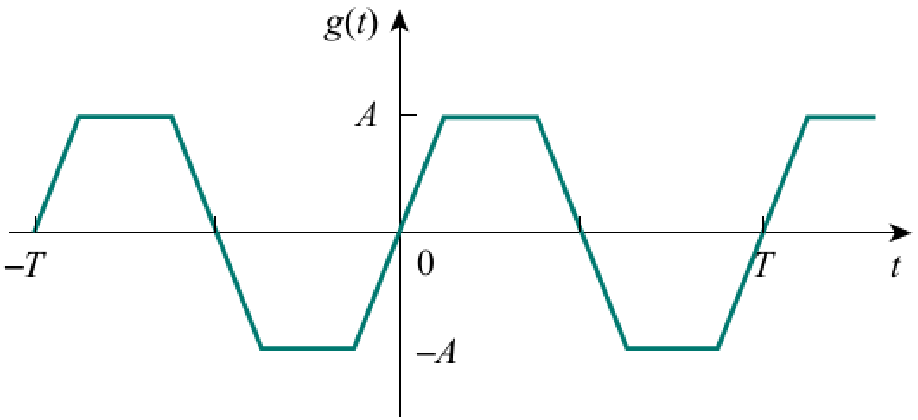
PERTIMBANGAN SIMETRI (3)

- Jika sinyal periodik tersebut diketahui simetrik ganjil setengah gelombang $f\left(t - \frac{T}{2}\right) = -f(t)$ maka diperoleh

$$a_0 = 0$$

$$a_n = \begin{cases} \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega_0 t\, dt, & \text{for } n \text{ odd} \\ 0, & \text{for } n \text{ even} \end{cases}$$

$$b_n = \begin{cases} \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega_0 t\, dt, & \text{for } n \text{ odd} \\ 0, & \text{for } n \text{ even} \end{cases}$$



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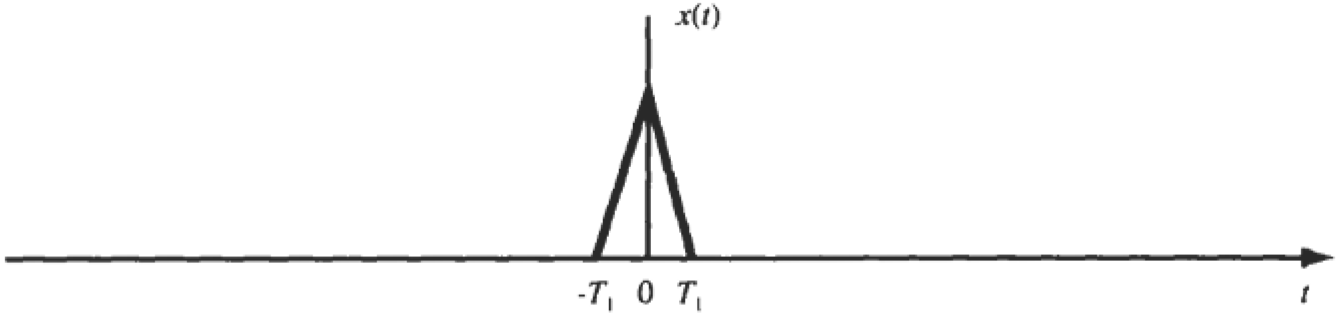
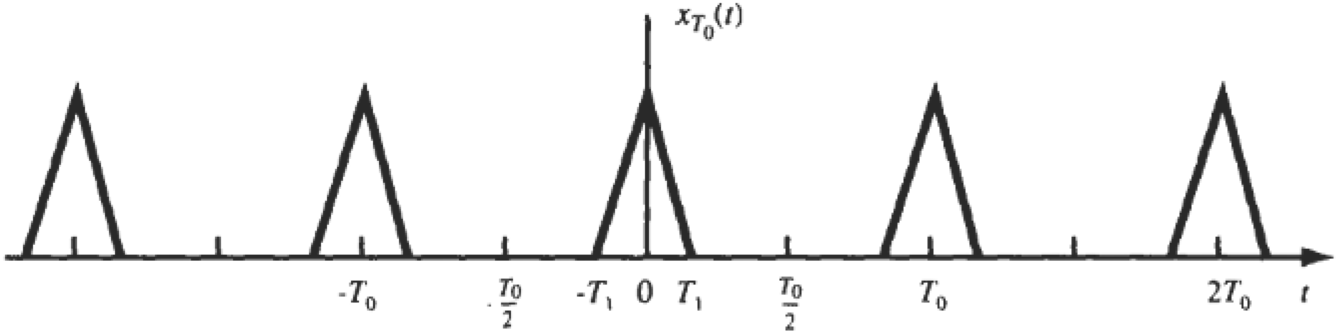
FROM FOURIER SERIES TO FOURIER TRANSFORM

- Let x(t) be a nonperiodic signal of finite duration, that is,
  $$x(t) = 0 \; ; |t| > T_1$$
- Such a signal is shown below . Let $x_{T0}(t)$ be a periodic signal formed by repeating x(r) with fundamental period T,



(a)



(b)

FROM FOURIER SERIES TO FOURIER TRANSFORM

- If we let $T_0 \to \infty$

$$\lim_{T_0 \to \infty} x_{T_0}(t) = x(t)$$

- The complex exponential Fourier series of $x_{T_0}(t)$ is given by

$$x_{T_0}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

where $c_k = \dfrac{1}{T_0}\displaystyle\int_{-T_0/2}^{T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt$

FROM FOURIER SERIES TO FOURIER TRANSFORM

- Since $x_{T_0}(t) = x(t)$ for $|t| < T_0/2$ and also since x(t) = 0 outside this interval

$$c_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T_0}\int_{-\infty}^{\infty} x(t)\, e^{-jk\omega_0 t}\, dt$$

- Let us define $X(\omega)$ as

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

FROM FOURIER SERIES TO FOURIER TRANSFORM

- the complex Fourier coefficients c_k can be expressed as

$$c_k = \frac{1}{T_0} X(k\omega_0)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

FOURIER TRANSFORM PAIR $x(t) \leftrightarrow X(\omega)$

- The function $X(\omega)$ is called the Fourier transform of $x(t)$

$$X(\omega) = \mathscr{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

- While the invers Fourier Transform is

$$x(t) = \mathscr{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

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PROPERTIES OF THE CONTINUOUS-TIME FOURIER TRANSFORM

1. Linearity

$$a_1 x_1(t) + a_2 x_2(t) \leftrightarrow a_1 X_1(\omega) + a_2 X_2(\omega)$$

2. Time Shifting

$$x(t - t_0) \leftrightarrow e^{-j\omega t_0} X(\omega)$$

the effect of a shift in the time domain is simply to add a linear term $-\omega t_0$, to the original phase spectrum $\theta(\omega)$. This is known as a linear phase shift of the Fourier transform $X(\omega)$

3. Frequency Shifting

The multiplication of x(t) by a complex exponential signal $e^{j\omega_0 t}$ is sometimes called complex modulation. Thus, Equation below shows that complex modulation in the time domain corresponds to a shift of $X(\omega)$ in the frequency domain.

$$e^{j\omega_0 t} x(t) \leftrightarrow X(\omega - \omega_0)$$

4. Time Scaling

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

- where a is a real constant.
- This equation indicates that scaling the time variable t by the factor a causes an inverse scaling of the frequency variable ω by $\frac{1}{a}$
- as well as an amplitude scaling of $X\left(\frac{\omega}{a}\right)$ by $\frac{1}{a}$ (. Thus, the scaling property implies that time compression of a signal ($a > 1$) results in its spectral expansion and that time expansion of the signal ($a < 1$) results in its spectral compression.

5. Time Reversal

time reversal of x(t) produces a like reversal of the frequency axis for $X(\omega)$. Equation below is readily obtained by setting $a = -1$

$$x(-t) \leftrightarrow X(-\omega)$$

6. Duality or symmetry

$$X(t) \leftrightarrow 2\pi x(-\omega)$$

7. Differentiation in the Time Domain

$$\frac{dx(t)}{dt} \leftrightarrow j\omega X(\omega)$$

8. Differentiation in the Frequency Domain

$$(-jt)x(t) \leftrightarrow \frac{dX(\omega)}{d\omega}$$

9. Integration in the time domain

$$\int_{-\infty}^{t} x(\tau)\,d\tau \leftrightarrow \pi X(0)\,\delta(\omega) + \frac{1}{j\omega}X(\omega)$$

10. Convolution

$$x_1(t) * x_2(t) \leftrightarrow X_1(\omega)X_2(\omega)$$

11. Multiplication

$$x_1(t)x_2(t) \leftrightarrow \frac{1}{2\pi}X_1(\omega) * X_2(\omega)$$

COMMON FOURIER TRANSFORM PAIRS

$x(t)$	$X(\omega)$
$\delta(t)$	1
$\delta(t-t_0)$	$e^{-j\omega t_0}$
1	$2\pi\delta(\omega)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
$\sin\omega_0 t$	$-j\pi[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$

$x(t)$	$X(\omega)$
$u(-t)$	$\pi\delta(\omega)-\frac{1}{j\omega}$
$e^{-at}u(t), a>0$	$\frac{1}{j\omega+a}$
$te^{-at}u(t), a>0$	$\frac{1}{(j\omega+a)^2}$
$e^{-a\|t\|}, a>0$	$\frac{2a}{a^2+\omega^2}$

COMMON FOURIER TRANSFORM PAIRS

$x(t)$	$X(\omega)$
$\dfrac{1}{a^2+t^2}$	$e^{-a\lvert\omega\rvert}$
$e^{-at^2}, a>0$	$\sqrt{\dfrac{\pi}{a}}\, e^{-\omega^2/4a}$
$p_a(t)=\begin{cases}1 & \lvert t\rvert<a\\ 0 & \lvert t\rvert>a\end{cases}$	$2a\dfrac{\sin\omega a}{\omega a}$

$x(t)$	$X(\omega)$
$\dfrac{\sin at}{\pi t}$	$p_a(\omega)=\begin{cases}1 & \lvert\omega\rvert<a\\ 0 & \lvert\omega\rvert>a\end{cases}$
$\operatorname{sgn} t$	$\dfrac{2}{j\omega}$
$\sum_{k=-\infty}^{\infty}\delta(t-kT)$	$\omega_0\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_0),\ \omega_0=\dfrac{2\pi}{T}$

Property	Signal	Fourier transform
	$x(t)$	$X(\omega)$
	$x_1(t)$	$X_1(\omega)$
	$x_2(t)$	$X_2(\omega)$
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(\omega)+a_2X_2(\omega)$
Time shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(\omega)$
Frequency shifting	$e^{j\omega_0 t}x(t)$	$X(\omega-\omega_0)$
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$
Time reversal	$x(-t)$	$X(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Time differentiation	$\frac{dx(t)}{dt}$	$j\omega X(\omega)$
Frequency differentiation	$(-jt)x(t)$	$\frac{dX(\omega)}{d\omega}$
Integration	$\int_{-\infty}^{t}x(\tau)\,d\tau$	$\pi X(0)\delta(\omega)+\frac{1}{j\omega}X(\omega)$
Convolution	$x_1(t)*x_2(t)$	$X_1(\omega)X_2(\omega)$
Multiplication	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1(\omega)*X_2(\omega)$
Real signal	$x(t)=x_e(t)+x_o(t)$	$X(\omega)=A(\omega)+jB(\omega)$ $X(-\omega)=X^*(\omega)$
Even component	$x_e(t)$	$\mathrm{Re}\{X(\omega)\}=A(\omega)$
Odd component	$x_o(t)$	$j\,\mathrm{Im}\{X(\omega)\}=jB(\omega)$
Parseval's relations		

$$\int_{-\infty}^{\infty}x_1(\lambda)X_2(\lambda)\,d\lambda=\int_{-\infty}^{\infty}X_1(\lambda)x_2(\lambda)\,d\lambda$$

$$\int_{-\infty}^{\infty}x_1(t)x_2(t)\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}X_1(\omega)X_2(-\omega)\,d\omega$$

$$\int_{-\infty}^{\infty}|x(t)|^2\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2\,d\omega$$

$x(t)$	$X(\omega)$
$\delta(t)$	1
$\delta(t-t_0)$	$e^{-j\omega t_0}$
1	$2\pi\delta(\omega)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
$\sin\omega_0 t$	$-j\pi[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
$u(-t)$	$\pi\delta(\omega)-\frac{1}{j\omega}$
$e^{-at}u(t),\ a>0$	$\frac{1}{j\omega+a}$
$te^{-at}u(t),\ a>0$	$\frac{1}{(j\omega+a)^2}$
$e^{-a\lvert t\rvert},\ a>0$	$\frac{2a}{a^2+\omega^2}$
$\frac{1}{a^2+t^2}$	$e^{-a\lvert\omega\rvert}$
$e^{-at^2},\ a>0$	$\sqrt{\frac{\pi}{a}}\,e^{-\omega^2/4a}$
$p_a(t)=\begin{cases}1 & \lvert t\rvert<a\\ 0 & \lvert t\rvert>a\end{cases}$	$2a\frac{\sin\omega a}{\omega a}$
$\frac{\sin at}{\pi t}$	$p_a(\omega)=\begin{cases}1 & \lvert\omega\rvert<a\\ 0 & \lvert\omega\rvert>a\end{cases}$
$\mathrm{sgn}\ t$	$\frac{2}{j\omega}$
$\sum_{k=-\infty}^{\infty}\delta(t-kT)$	$\omega_0\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_0),\ \omega_0=\frac{2\pi}{T}$

FREQUENCY RESPONSE

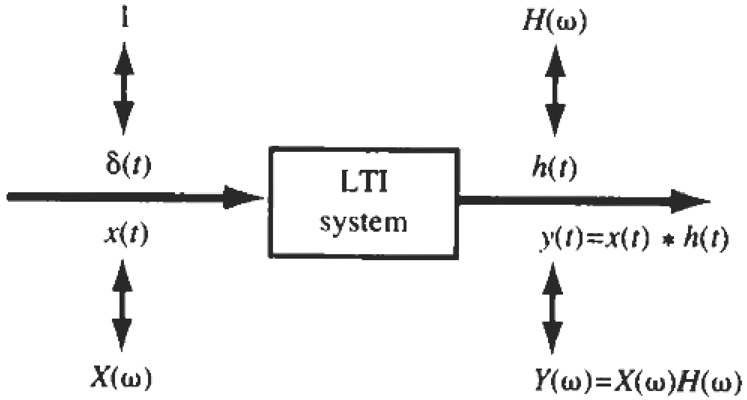
- The output $y(t)$ of a continuous-time LTI system equals the convolution of the input $x(t)$ with the impulse response $h(t)$

$$y(t) = x(t) * h(t)$$

- Applying the convolution property, we obtain:

$$Y(\omega) = X(\omega)H(\omega)$$

- Relationships between inputs and outputs in an LTI system shown by:



FREQUENCY RESPONSE

- Where $Y(\omega), X(\omega)$ and $H(\omega)$ are the Fourier transforms of $y(t), x(t)$ and $h(t)$, respectively we have

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

- The function $H(\omega)$ is called the frequency response of the system.
- Let $H(\omega) = |H(\omega)|e^{j\theta_H(\omega)}$
- Then $|H(\omega)|$ is called the magnitude response of the system
- And $\theta_H(\omega)$ is called the phase response of the system

$$\theta_H(\omega) = \tan^{-1} \omega$$

FREQUENCY RESPONSE

A system described by: $y'(t) + 2y(t) = x(t) + x'(t)$

Find the impulse response $h(t)$ of the system

Answer
$j\omega Y(\omega) + 2Y(\omega) = X(\omega) + j\omega X(\omega)$

Or $(j\omega + 2)Y(\omega) = (1 + j\omega)X(\omega)$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1 + j\omega}{2 + j\omega} = \frac{2 + j\omega - 1}{2 + j\omega} = 1 - \frac{1}{2 + j\omega}$$

Inverse Fourier Transform of $H(\omega)$, the impulse response $h(t)$ is

$$h(t) = \delta(t) - e^{-2t}u(t)$$

$$\frac{dy(t)}{dx} + 2y(t) = x(t)$$

Find the output $y(t)$ if $x(t) = e^{-1}u(t)$

Answer:

$$j\omega Y(\omega) + 2Y(\omega) = X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega} = \frac{1}{2 + j\omega}$$

$$e^{-1}u(t) \quad \leftrightarrow \quad \frac{1}{1+j\omega}$$

$$Y(\omega) = \mathrm{X}(\omega)\ \mathrm{H}(\omega) = \frac{1}{(1+j\omega)(2+j\omega)} = \frac{1}{1+j\omega} - \frac{1}{2+j\omega} \leftrightarrow y(t) = (e^{-t} - e^{-2t})u(t)$$

DISCRETE FOURIER TRANSFORM

DISCRETE FOURIER TRANSFORM $x[n] \leftrightarrow X(\Omega)$

$$X(\Omega) = \mathscr{F}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

$$x[n] = \mathscr{F}^{-1}\{X(\Omega)\} = \frac{1}{2\pi} \int_{2\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega$$

Table 6-1. Properties of the Fourier Transform

Property	Sequence	Fourier transform
	$x[n]$	$X(\Omega)$
	$x_1[n]$	$X_1(\Omega)$
	$x_2[n]$	$X_2(\Omega)$
Periodicity	$x[n]$	$X(\Omega+2\pi)=X(\Omega)$
Linearity	$a_1x_1[n]+a_2x_2[n]$	$a_1X_1(\Omega)+a_2X_2(\Omega)$
Time shifting	$x[n-n_0]$	$e^{-j\Omega n_0}X(\Omega)$
Frequency shifting	$e^{j\Omega_0 n}x[n]$	$X(\Omega-\Omega_0)$
Conjugation	$x^*[n]$	$X^*(-\Omega)$
Time reversal	$x[-n]$	$X(-\Omega)$
Time scaling	$x_{(m)}[n]=\begin{cases} x[n/m] & \text{if } n=km \\ 0 & \text{if } n\neq km\end{cases}$	$X(m\Omega)$
Frequency differentiation	$nx[n]$	$j\dfrac{dX(\Omega)}{d\Omega}$
First difference	$x[n]-x[n-1]$	$(1-e^{-j\Omega})X(\Omega)$
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\pi X(0)\delta(\Omega)+\dfrac{1}{1-e^{-j\Omega}}X(\Omega)$ $\lvert\Omega\rvert\leq\pi$
Convolution	$x_1[n]*x_2[n]$	$X_1(\Omega)X_2(\Omega)$
Multiplication	$x_1[n]x_2[n]$	$\dfrac{1}{2\pi}X_1(\Omega)\otimes X_2(\Omega)$
Real sequence	$x[n]=x_e[n]+x_o[n]$	$X(\Omega)=A(\Omega)+jB(\Omega)$ $X(-\Omega)=X^*(\Omega)$
Even component	$x_e[n]$	$\text{Re}\{X(\Omega)\}=A(\Omega)$
Odd component	$x_o[n]$	$j\,\text{Im}\{X(\Omega)\}=jB(\Omega)$
Parseval's relations		

$$\sum_{n=-\infty}^{\infty} x_1[n]x_2[n]=\frac{1}{2\pi}\int_{2\pi} X_1(\Omega)X_2(-\Omega)\,d\Omega$$

$$\sum_{n=-\infty}^{\infty} |x[n]|^2=\frac{1}{2\pi}\int_{2\pi} |X(\Omega)|^2\,d\Omega$$

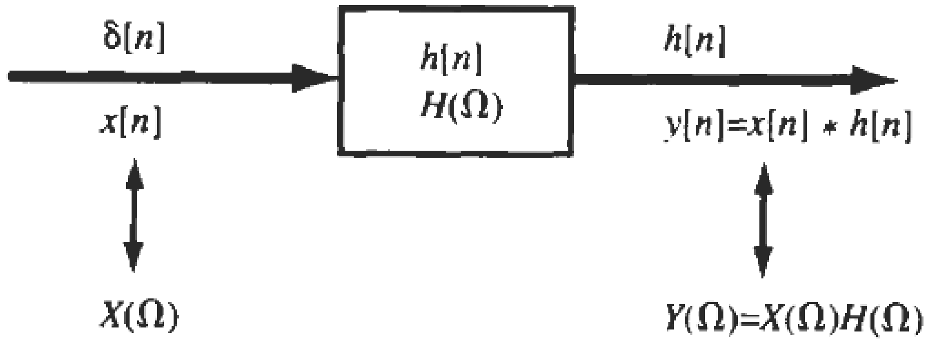
$x[n]$	$X(\Omega)$
$\delta[n]$	1
$\delta[n-n_0]$	$e^{-j\Omega n_0}$
$x[n]=1$	$2\pi\delta(\Omega), \lvert\Omega\rvert\leq\pi$
$e^{j\Omega_0 n}$	$2\pi\delta(\Omega-\Omega_0), \lvert\Omega\rvert, \lvert\Omega_0\rvert\leq\pi$
$\cos\Omega_0 n$	$\pi[\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)], \lvert\Omega\rvert, \lvert\Omega_0\rvert\leq\pi$
$\sin\Omega_0 n$	$-j\pi[\delta(\Omega-\Omega_0)-\delta(\Omega+\Omega_0)], \lvert\Omega\rvert, \lvert\Omega_0\rvert\leq\pi$
$u[n]$	$\pi\delta(\Omega)+\dfrac{1}{1-e^{-j\Omega}}, \lvert\Omega\rvert\leq\pi$
$-u[-n-1]$	$-\pi\delta(\Omega)+\dfrac{1}{1-e^{-j\Omega}}, \lvert\Omega\rvert\leq\pi$
$a^n u[n], \lvert a\rvert<1$	$\dfrac{1}{1-ae^{-j\Omega}}$
$-a^n u[-n-1], \lvert a\rvert>1$	$\dfrac{1}{1-ae^{-j\Omega}}$
$(n+1)a^n u[n], \lvert a\rvert<1$	$\dfrac{1}{(1-ae^{-j\Omega})^2}$
$a^{\lvert n\rvert}, \lvert a\rvert<1$	$\dfrac{1-a^2}{1-2a\cos\Omega+a^2}$
$x[n]=\begin{cases}1 & \lvert n\rvert\leq N_1 \\ 0 & \lvert n\rvert>N_1\end{cases}$	$\dfrac{\sin[\Omega(N_1+\frac{1}{2})]}{\sin(\Omega/2)}$
$\dfrac{\sin Wn}{\pi n}, 0<W<\pi$	$X(\Omega)=\begin{cases}1 & 0\leq\lvert\Omega\rvert\leq W \\ 0 & W<\lvert\Omega\rvert\leq\pi\end{cases}$
$\sum_{k=-\infty}^{\infty}\delta[n-kN_0]$	$\Omega_0\sum_{k=-\infty}^{\infty}\delta(\Omega-k\Omega_0), \Omega_0=\dfrac{2\pi}{N_0}$

FREQUENCY RESPONSE

- $y[n] = x[n] * h[n] \leftrightarrow Y(\Omega) = X(\Omega)H(\Omega)$

- $H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} \quad \rightarrow \quad H(\Omega) = |H(\Omega)|e^{-j\theta_H(\Omega)}$

- The relationship is figured by:



As in the continuous-time case, the function $H(\Omega)$ is called the frequency response of the system. $|H(\Omega)|$ the magnitude response and $\theta_H(\Omega)$ is the phase response

- A causal discrete time LTI system is described by:

$$y[n]-\frac{3}{4}y[n-1]+\frac{1}{8}y[n-2]=x[n]$$

a. Determine the frequency response $H(\Omega)$

b. Find the impulse response $h[n]$

$$Y(\Omega)-\frac{3}{4}e^{-j\Omega}Y(\Omega)+\frac{1}{8}e^{-j2\Omega}Y(\Omega)=X(\Omega)$$

$$\left(1-\frac{3}{4}e^{-j\Omega}+\frac{1}{8}e^{-j2\Omega}\right)Y(\Omega)=X(\Omega)$$

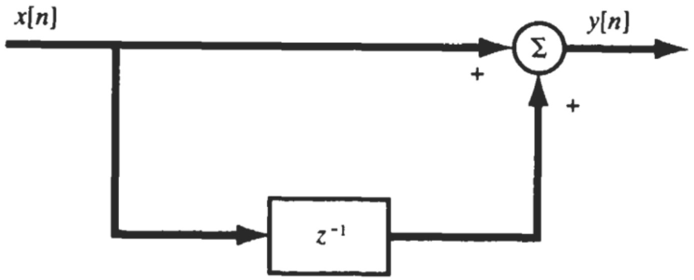
$$H(\Omega)=\frac{1}{\left(1-\frac{1}{2}e^{-j\Omega}\right)\left(1-\frac{1}{4}e^{-j\Omega}\right)}\leftrightarrow\frac{2}{\left(1-\frac{1}{2}e^{-j\Omega}\right)}-\frac{1}{\left(1-\frac{1}{4}e^{-j\Omega}\right)}\quad h[n]=\left[2\left(\frac{1}{2}\right)^{n}-\left(\frac{1}{4}\right)^{n}\right]u[n]$$

- $y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{2}x[n-1]$
- **Determine** $H(\Omega), h[n]$

$$Y(\Omega) - \frac{1}{2}e^{-j\Omega}Y(\Omega) = X(\Omega) + \frac{1}{2}e^{-j\Omega}X(\Omega)$$

$$H(\Omega) = \frac{1+\frac{1}{2}e^{-j\Omega}}{1-\frac{1}{2}e^{-j\Omega}} = \frac{1}{1-\frac{1}{2}e^{-j\Omega}} + \frac{1}{2}\frac{e^{-j\Omega}}{1-\frac{1}{2}e^{-j\Omega}}$$

$$h[n] = \left(\frac{1}{2}\right)^n u[n] + \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} u[n-1]$$



(*a*) Find the frequency response $H(\Omega)$ of the system.
(*b*) Find the impulse response $h[n]$ of the system.
(*c*) Sketch the magnitude response $|H(\Omega)|$ and the phase response $\theta(\Omega)$.

$$y[n] = x[n] + x[n-1]$$

$$h[n] = \delta[n] + \delta[n-1]$$

$$|H(\Omega)| = 2\cos\left(\frac{\Omega}{2}\right)$$

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)}$$

$$\theta(\Omega) = -\frac{\Omega}{2}$$

$$= 1 + e^{-j\Omega}$$

$$= e^{-j\Omega/2}\left(e^{j\Omega/2} + e^{-j\Omega/2}\right)$$

$$= 2e^{-j\Omega/2}\cos\left(\frac{\Omega}{2}\right)$$

PLOT