LINEAR TIME-INVARIANT SYSTEM

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INTRODUCTION

- Two most important attributes of systems are:

1. Linearity
2. Time-invariance.

- The input-output relationship for LTI systems is described in terms of a convolution operation.
- The importance of the convolution operation in LTI systems stems from the fact that knowledge of the response of an LTI system to the unit impulse input allows us to find its output to any input signals.

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RESPONSE OF A CONTINUOUS-TIME LTI SYSTEM AND THE CONVOLUTION INTEGRAL

IMPULSE RESPONSE

- The impulse response $h(t)$ of a continuous-time LTI system (represented by T) is defined to be the response of the system when the input is $\delta(t)$

$$h(t) = T\{\delta(t)\}$$

- The input $x(t)$ can be expressed as:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

- Since the system is linear, the response $y(t)$ of the system to an arbitrary input $x(t)$ can be expressed as

$$y(t) = T\{x(t)\} = T\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right\}$$

$$= \int_{-\infty}^{\infty} x(\tau)T\{\delta(t-\tau)\}d\tau$$

IMPULSE RESPONSE

- Since the system is time-invariant, we have:

$$h(t-\tau) = T\{\delta(t-\tau)\}$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

- This equation indicates that a continuous-time LTI system is completely characterized by its impulse response h(t)

CONVOLUTION INTEGRAL

- Convolution of two continuous-time signals $x(t)$ and $h(t)$ denoted by:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

- This equation is commonly called the convolution integral.
- Thus, we have the fundamental result that the output of any continuous-time LTI system is the convolution of the input **x(t)** with the impulse response **h(t)** of the system.

PROPERTIES OF CONVOLUTION INTEGRAL

- Commutative

$$x(t) * h(t) = h(t) * x(t)$$

- Associative

$$\{x(t) * h_1(t)\} * h_2(t) = x(t) * \{h_1(t)*h_2(t)\}$$

- Distributive

$$x(t) * \{h_1(t) + h_2(t)\} = x(t) * h_1(t) + x(t)*h_2(t)\}$$

CONVOLUTION INTEGRAL OPERATION

- Applying the commutative property, we can obtain

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \quad \Rightarrow \quad y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

- Observe that the convolution integral operation involves the following four steps:
  1. The impulse response $\mathrm{h}(\tau)$ is time-reversed (that is, reflected about the origin) to obtain $h(-\tau)$ and then shifted by t to form $h(t-\tau) = h(-(\tau - t))$which is a function of τ with parameter t
  2. The signal $x(\tau)$ *and* $h(t-\tau)$ *are multiplied together for all values of* τ *with t fixed at* some value
  3. The product $x(\tau)h(t-\tau)$ is integrated over all τ to produce a single output value $y(t)$
  4. Steps 1 to 3 are repeated as t varies over $-\infty$ *to* ∞ *to produce the entire output* $y(t)$.

STEP RESPON

- The step response s(t) of a continuous-time LTI system (represented by T) is defined to be the response of the system when the input is u(t)

$$s(t) = T\{u(t)\}$$

- In many applications, the step response s(t) is also a useful characterization of the system. The step response s(t) can be easily determined

$$s(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} h(\tau)d\tau$$

- Thus, the step response s(t) can be obtained by integrating the impulse response h(t). And vise versa

$$h(t) = s'(t) = \frac{ds(t)}{dt}$$

PROPERTIES OF CONTINUOUS-TIME LTI SYSTEMS

SYSTEM WITH OR WITHOUT MEMORY

- Since the output y(t) of a memoryless system depends on only the present input x(t), then, if the system is also linear and time-invariant, this relationship can only be of the form

$$y(t) = Kx(t)$$

- Where K is a (gain) constant. Thus the corresponding impulse response h(t) is simply:

$$h(t) = K\delta(t)$$

- Therefore, if $h(t_0) \neq 0$ for $t_0 \neq 0$, the continuous-time LTI system has memory

CAUSALITY

- A causal system does not respond to an input event until that event actually occurs. Therefore, for a causal continuous-time LTI system, we have

$$h(t) = 0 \qquad\qquad t < 0$$

- Applying the causality condition, to equation

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

The output of a causal continuous-time LTI system is expressed as:

$$y(t) = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau$$

Or

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau$$

CAUSALITY

- Based on causality condition $h(t) = 0\ ;\quad t < 0$ any signal $x(t)$ is called causal if:

$$x(t) = 0\ ;\quad t < 0$$

- and is called anticausal if

$$x(t) = 0\ ;\quad t > 0$$

When the input $x(t)$ is causal, the output $y(t)$ of a causal continuous-time LTI system is given by:

$$y(t) = \int_0^t h(\tau)x(t-\tau)d\tau = \int_0^t x(\tau)h(t-\tau)d\tau$$

STABILITY

- The BIBO (bounded-input/bounded-output) stability of an LTI system is readily ascertained from its impulse response.
- It can be shown (in step response section) that a continuous-time LTI system is BIBO stable if its impulse response is absolutely integrable, that is

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$$

RESPONSE OF A DISCRETE-TIME LTI SYSTEM AND CONVOLUTION SUM

IMPULSE RESPONSE

- The impulse response (or unit sample response) $h[n]$ ***of a discrete-time LTI system*** (represented by **T) is defined to be the response of the system when the input is $\delta[n]$, that** is,

$$h[n] = T\{\delta[n]\}$$

- The input $x[n]$ can be expressed as:

$$x[n] = \sum_{-\infty}^{\infty} x[k]\delta[n-k]$$

- Since the system is linear, the response ***y[n] of the system to an arbitrary input x[n] can be*** expressed as

$$y[n] = T\{x[n]\} = T\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\}$$

$$= \sum_{k=-\infty}^{\infty} x[k]T\{\delta[n-k]\}$$

- Since the system is time invariant, we have:

$$h[n-k] = T\{\delta[n-k]\}$$

- So we can obtain:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

CONVOLUTION SUM

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- This equation defines the convolution of two sequences x[n] and h[n] denoted by:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- commonly called the convolution sum. Thus, again, we have the fundamental result that the output of any discrete-time LTI system is the convolution of the input x[n] with the impulse response h[n] of the system.

PROPERTIES OF THE CONVOLUTION SUM

- Commutative

$$x[n] * h[n] = h[n] * x[n]$$

- Associative

$$\{x[n] * h_1[n]\} * h_2[n] = x[n] * \{h_1[n] * h_2[n]\}$$

- Distributive

$$x[n] * \{h_1[n] + h_2[n]\} = x[n] * h_1[n] + x[n] * h_2[n]$$

CONVOLUTION SUM OPERATION

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Similar to the continuous-time case:

1. The impulse response h[k] is time-reversed (that is, reflected about the origin) to obtain h[-k] and then shifted by n to form h[n - k] = h[-(k - n)] which is a function of k with parameter n.
2. Two sequences x[k] and h[n - k] are multiplied together for all values of k with n fixed at some value.
3. The product x[k]h[n - k] is summed over all k to produce a single output sample y[n]
4. Steps 1 to 3 are repeated as n varies over -∞ to ∞ to produce the entire output y[n].

STEP RESPONSE

- The *step response s[n] of a discrete-time LTI system with the impulse response h[n] is:*

$$\mathrm{s[n]} = h\mathrm{[n]} * u\mathrm{[n]} = \sum_{\mathrm{k}=-\infty}^{\infty} \mathrm{h[k]u[n-k]} = \sum_{k=-\infty}^{n} h[k]$$

$$h[n] = s[n] - s[n-1]$$